Acoustic Instabilities in Nonadiabatic Weakly Ionized Gases with Applied Magnetic Fields

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Introduction

THE problem of sound generation in air flowing through a heated screen was considered first by Lord Rayleigh,[1] who did not take into account, however, the presence of dissipative processes such as momentum losses due to viscosity and energy transfer by heat conduction and radiation. This was done more recently by Glushkov and Kareev,[2] who showed that the Rayleigh criterion is a necessary but not a sufficient condition for the excitation of disturbances in the gas. Furthermore, they determined the condition for sufficiency and derived a formula for the characteristic dimension of a stable flame cell.

It seems of interest to attempt an extension of this problem to include the case of nonadiabatic weakly ionized gases in the presence of applied magnetic fields. The present study concerns a weakly ionized gaseous system under the influence of a magnetic field $\boldsymbol{B}$ applied normal to the direction of propagation of the disturbances. Such disturbances can be seen as pressure or density increments caused by thermal source oscillations arising in the system. The influence of $\boldsymbol{B}$ on the characteristic dimension of a stable cell also is examined. The technique used in the mathematical approach is the one used by Glushkov and Kareev,[2] including, however, the appropriate electromagnetic terms and equations for this problem.

Basic Equations and Assumptions

Consider a weakly ionized nonatomic gas in the presence of an externally applied magnetic field $\boldsymbol{B}$. Assuming further that the MHD approximations hold,[3] one can write the flow equations as follows:

Mass

$$(\mathrm{D}\rho/\mathrm{D}t) + (\partial v_k/\partial x_k) = 0 \tag{1}$$

Momentum

$$\rho\frac{\mathrm{D}v_i}{\mathrm{D}t} + \frac{\partial}{\partial x_k}\left[P + \frac{B^2}{2\mu}\right]\delta_{ik} - \frac{\partial}{\partial x_k}\left[\frac{B_iB_k}{\mu}\right] - \frac{4}{3}\eta\frac{\partial}{\partial x_i} \times\left[\frac{\partial v_k}{\partial x_k}\right] + \eta\frac{\partial}{\partial x_k}\left[\frac{\partial v_k}{\partial x_i}\right] - \eta\frac{\partial}{\partial x_k}\left[\frac{\partial v_i}{\partial x_k}\right] \tag{2}$$

State

$$P(k/M)\rho T = 0 \tag{3}$$

Energy

$$\mathrm{D}p/\mathrm{D}t = a_s^2(\mathrm{D}\rho/\mathrm{D}t) + (\gamma-1)\rho(\mathrm{D}Q/\mathrm{D}t) \tag{4}$$

Induction

$$\frac{\partial B_i}{\partial t} = \frac{\partial}{\partial x_k}\left[v_iB_k - v_kB_i\right] - \frac{1}{\mu\sigma}\left[\frac{\partial B_i}{\partial x_k} - \frac{\partial B_k}{\partial x_i}\right] = 0 \tag{5}$$

Furthermore, the amount of heat evolved per unit volume in unit time can be written as[2]

$$\rho\frac{\mathrm{D}Q}{\mathrm{D}t} = q_c + k\frac{\partial}{\partial x_k}\left(\frac{\partial T}{\partial x_k}\right) - \frac{\epsilon\tau}{\delta}T^4 - (\epsilon_{ijk}\sigma v_j B_k)^2 + \frac{4}{3}\eta v_i\frac{\partial}{\partial x_i}\left(\frac{\partial v_k}{\partial x_k}\right) + \eta v_i\frac{\partial}{\partial x_k}\left(\frac{\partial v_k}{\partial x_i}\right) - \eta v_i\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_k}\right) = 0 \tag{6}$$

In Eqs. (1-6), ρ, V, p, B, and T are the density, velocity, pressure, applied magnetic field, and temperature, respectively; η and δ are the transport coefficients (viscosity and electrical conductivity), k is the Boltzmann constant, M the reduced mass, γ the specific heat ratio, μ the magnitude permeability, and a_s the isentropic speed of sound in the gas; q_c is the heat source strength per unit volume of the gas, ϵ the gas emissivity, τ the Stephan-Boltzmann constant, k the thermal conductivity, and δ the thickness of a gas layer assumed to be plane.

Solution of the Problem

If the gas is stationary (at rest, all gradients zero), one can write, for small disturbances in the system§ from Eq. (1),

$$\partial\rho'/\partial t = -\rho(\partial v_k'/\partial x_k) \tag{7}$$

From Eq. (2),

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i}\left[\frac{p'}{\rho}\right] + \frac{B_k}{\mu\rho}\frac{\partial}{\partial x_k}\left[\frac{\partial B_k'}{\partial x_i} - \frac{\partial B_i'}{\partial x_k}\right] - \frac{4}{3}v\frac{\partial}{\partial x_i}\left[\frac{\partial v_k'}{\partial x_k}\right] + v\frac{\partial}{\partial \chi_k}\left[\frac{\partial v_k'}{\partial x_i}\right] - v\frac{\partial}{\partial x_k}\left[\frac{\partial v_i'}{\partial x_k}\right] = 0 \tag{8}$$

From Eq. (3),

$$p'/p = (\rho'/\rho) + (T'/T) \tag{9}$$

From Eq. (4), after substituting Eq. (6) into it,

$$\frac{\partial p'}{\partial t} - a_s^2\frac{\partial\rho'}{\partial t} - (\gamma-1)\left[q_c' + k\frac{\partial^2 T'}{\partial x_k 2} - 4\frac{\epsilon\tau}{\delta}T^3T'\right] = 0 \tag{10}$$

and, from Eq. (5),

$$\frac{\partial B_i'}{\partial t} + \frac{\partial}{\partial x_k}[B_iv_k' - B_kv_i'] - \frac{1}{\mu\sigma}\frac{\partial}{\partial x_k}\left[\frac{\partial B_i'}{\partial x_k} - \frac{\partial B_k'}{\partial x_i}\right] = 0 \tag{11}$$

where $v = \eta/\delta$, and $\lambda = 2\delta$.

Assuming that $q_c = q_c\ (p,\rho)$, one also can write

$$q_c' = \left(\frac{\partial q_c}{\partial p}\right)_\rho (p' - a_{q_c}^2\rho') \tag{12}$$

where $a_{q_c}^2 = (\partial p/\partial\rho)_{q_c}$.

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§In general, $\bar{C}_{ij} = \bar{C}_{ij} + C_{ij}'$, where $\bar{C}_{ij}$ is the undisturbed parameter and $|C_{ij}'| << |\bar{C}_{ij}|$. Notice, however, that, from Eq. (7) on, the bar above undisturbed values has been omitted.

Assuming further that the disturbances are proportional to $\exp(i\boldsymbol{k}\cdot\boldsymbol{r}-iwt)$ and choosing the magnetosonic frequency to be the medium characteristic frequency (that is, $w=ka$, where k is the magnetosonic wave number and a the magnetosonic speed¶), one has the following set of equations:

$$w\rho' = \rho k_k v_k' \tag{13}$$

$$wv_i' - k_i(p'/\rho) - (B_k/\mu\rho)[k_i B_k' - k_k B_i'] + ik_i k_k 4/3\nu v_k'$$
$$- ik_j k_i \nu v_j' + ik^2 \nu v_i' = 0 \tag{14}$$

$$p'/p = (\rho'/\rho) + (T'/T)wP' - a_s^2 w\rho'$$
$$- i(\gamma-1)[q_{c'} - \kappa k^2 T' - 4q_r(T'/T)] = 0 \tag{15}$$

$$q_{c'} = \left[\frac{\partial q_c}{\partial P}\right]_\rho p' + \left[\frac{\partial q_c}{\partial \rho}\right]_p \rho' \quad wB_i' - k_k[B_i' v_k' - B_k v_i']$$
$$+ i\frac{k_k}{\mu\sigma}[k_k B_i' - k_i B_k'] = 0 \tag{16}$$

Choosing the x_1 direction to be the one in which the disturbance is propagated and assuming that the $\boldsymbol{B}$ field is applied along the x_3 direction (i.e., $\boldsymbol{B}=B_3\epsilon_3$) and using the fact that[2] $(a_{qc}/a_s)^2 \simeq 1/\gamma$, Eqs. (13-16) yield, after some algebra,

$$(1-\phi\xi_B{}^2)y^3 - i[\phi\xi_B{}^2(\xi_T+\xi_r-\xi_c)z + (\xi_c-\xi_T-\xi_r)$$
$$\times z - \phi\xi_B\theta - \xi_v z]y^2 - [z^2 - \phi\xi_B(\xi_c-\xi_T-\xi_r)z\theta - \xi_\nu$$
$$\times(\xi_c-\xi_T-\xi_r)z^2]y - i[(\xi_T+\xi_r-\xi_c)/\gamma]z^3 = 0 \tag{17}$$

where**

$$y = \frac{w}{ka}, \qquad \xi_r = 4\frac{\gamma-1}{k_s a_s}\frac{\epsilon\tau T^4}{\delta P}$$

$$\xi_B = \frac{\mu+\sigma+b}{k_s}, \qquad \phi = \frac{k^4}{(\mu\sigma w)^2 + k^4}$$

$$\xi_\nu = \frac{4}{3}\nu\frac{k_s^2}{k_s a_s}, \qquad z = \frac{a_s}{a}$$

$$\xi_c = \frac{\gamma-1}{k_s a_s}\left(\frac{\partial q_c}{\partial P}\right)_\rho, \qquad \theta = \frac{b}{a}$$

$$\xi_T = \frac{\gamma-1}{k_s a_s}\kappa k_s{}^2\frac{T}{P}$$

Solutions to (17) can be found easily if one assumes not only that[2] $\xi_\nu\xi << 1$ but further that $\xi_\nu\xi(\lambda\boldsymbol{B}/2\pi)\sqrt{\eta/\sigma} << 1$, where $\xi=\xi_c-\xi_T-\xi_r$. This assumption imposes a limiting size on the gas-layer thickness δ, since it implies that $\delta \leq \pi\sqrt{\eta/\sigma}/\boldsymbol{B}$.

Then Eq. (17) becomes

$$(1-\phi\xi_B^2)y^3 - i[(\xi-\xi_V)z - \phi\xi_B\theta - \phi\xi\xi\xi_B^2 z]$$
$$\times y^2 - z^2 y + i(\xi/\gamma)z^3 = 0 \tag{18}$$

From Eq. (18), periodic and nonperiodic solutions can be obtained by substituting $y_r - iy_i$ and $-iy_i$, respectively, for y.

¶Recall that $a=\sqrt{a_s^2+b^2}$, where $b=B_3/\sqrt{\rho\mu}$ is the Afven velocity.

**Notice that k_s is the acoustic wave number.

Hence,

$$y_i{}^3 + \left[\frac{(\xi-\xi_\nu)z - \phi\xi_B\theta - \phi\xi\,\xi_B{}^2\theta^2 z}{1-\phi\xi_B{}^2}\right]y_i{}^2$$
$$+ \left[\frac{z^2}{1-\phi\xi_B{}^2}\right]y_i + \left[\frac{\xi z^3}{\gamma(1-\phi\xi_B{}^2)}\right] = 0$$

for the aperiodic solution, and

$$y_i{}^3 + \left[\frac{(\xi-\xi_\nu)z - \phi\xi_B\theta - \phi\xi\,\xi_B{}^2\theta^2 z}{1-\phi\xi_B{}^2}\right]y_i{}^2$$
$$+ \frac{1}{4}\left\{\left[\frac{(\xi-\xi_\nu)z - \phi\xi_B\theta - \phi\xi\,\xi_B^2\theta^2 z}{1-\phi\xi_B{}^2}\right]^2 + \left[\frac{z^2}{1-\phi\xi_B{}^2}\right]\right\}y_i$$
$$+ \frac{1}{8}\left\{\left[\frac{(\xi-\xi_\nu)z^3 - \phi\xi_B z^2\theta - \phi\nu\theta^2\xi_B{}^2 z^3}{(1-\phi\xi_B^2)^2}\right]\right.$$
$$\left. - \left[\frac{\xi z^3}{\gamma(1-\phi\xi_B^2)}\right]\right\} = 0 \tag{20}$$

for the periodic one.

Notice that Eqs. (21) and (22) collapse into the Glushkov and Kareev[2] results if $\boldsymbol{B}=0$ or $\sigma=0$, as expected. Equations (21) and (22) are shown in Fig. 1, along with the results of Ref. 2. One can calculate the value of ξ_c for the nonperiodic solution stability condition to find

$$\xi_c = (\xi^*_{cnp})_{B,\,\sigma\neq0} = (\xi^*_{cnp})_{B \text{ or } \tau=0} = \xi_T + \xi_r \text{ same as in Ref. 2} \tag{21}$$

However, when this is done for the periodic solution, one has

$$\xi_c = (\xi^*_{cp})_{B,\,\sigma\neq0} = (\xi^*_{cnp}) + \left(\frac{\gamma}{\gamma-1-\gamma\xi_B^2}\right)\xi_\nu$$
$$+ \left(\frac{\gamma}{\gamma-1-\gamma\xi_B^2}\right)\left(\frac{b}{a_s}\right)\xi_B \tag{22}$$

thus presenting the shift shown in Fig. 1 when compared with the results of Glushkov and Kareev.[2]

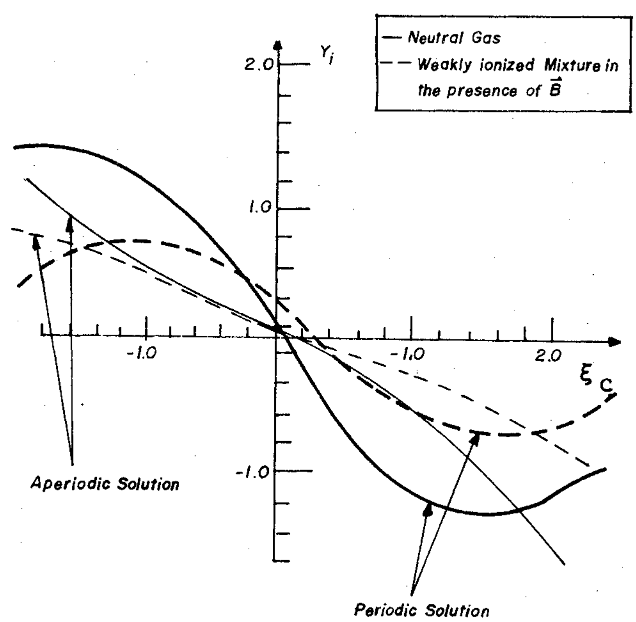

Fig. 1 Mixture of argon -0.179% cesium; $B=10$ kG, $p=1$ atm.

Discussion of the Results

Solutions of Eq. (18) allow a description of the disturbances arising in a weakly ionized gas containing heat sources in the presence of an applied normal magnetic field. These solutions can be compared with those obtained by Glushkov and Kareev[2] for a neutral gas.

As before, the condition for generating disturbances in the gas can be written as

$$\left(\frac{\partial q_c}{\partial p}\right)_\rho < \frac{\kappa_s a_s}{\gamma - 1}\xi_c^* > 0 \tag{23}$$

where ξ_c^* is determined from (19) and (20).

Notice that, although ξ_{cnp}^* does not change whether or not there is an applied $\boldsymbol{B}$ field, this is not the case for ξ_{cp}^* (Fig. 1). Actually, there exists a shift toward greater values of ξ_c if an ionized gas with an applied normal $\boldsymbol{B}$ field is considered. As shown by Glushkov and Kareev,[2] if condition (23) is met, fluctuations in the strength of heat sources can lead to both wave and aperiodic disturbances. However, one can see that the presence of the magnetic field causes a reduction in the values of $|y_i|$ for the same ξ_c.

As the criterion for absolute stability still is given by the aperiodic stability limit, which is almost insensitive to the $\boldsymbol{B}$ field, one may conclude that the characteristic dimension of an absolutely stable system again can be determined by the relationship suggested by Glushkov and Kareev.[2]

References

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Light Reflection as a Simple Experimental Method in Compressible Boundary-Layer Studies

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Introduction

In compressible flow investigations optical methods are extensively applied and offer a wide variety of possibilities. This paper discusses the application of light reflection as an experimental method to measure the local refractive index of a gas at the interface with a wall, even if rather strong gradients in temperature, density, or chemical composition of the gas exist. Such a situation is met in a compressible boundary layer adjacent to a wall. Let us assume that the wall is optically transparent and that a plane light wave is reflecting from the interface between wall and gas and from the boundary layer. It will be shown that if the ratio of the boundary layer thickness to the wavelength is not too small the reflections from the boundary layer itself vanish, and the refractive index of the gas at the wall is obtained by a measurement of the reflectivity.

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The refractive index of a gas depends on its density and chemical composition. Possible fields of application of the method are, therefore, mixing and diffusion processes, temperature jump, and chemical reaction phenomena in compressible boundary layers. Some applications of the method to the viscous side-wall and the thermal end-wall boundary layer in a shock tube are briefly discussed.

Theory

Consider a gaseous compressible boundary layer adjacent to an optically transparent wall. A plane light wave, linearly polarized, reflects partly from the plane of separation between the two media, partly from the gaseous inhomogeneity as indicated in Fig. 1. Subscript a will be used for the solid, s to indicate the gas state at $x=0$. The gas refractive index varies from a value n_s at the wall to n_∞ as $x \to \infty$. The angle between the direction of light propagation and the x direction is denoted by θ. A detailed discussion on light reflection from stratified media has been given by Jacobsson.[1]

We have to distinguish two different reflection phenomena, the direct reflection from the discontinuity and internal reflections from the boundary layer.

Direct Reflection from Discontinuity

The amplitude reflection coefficient for the direct reflection is given by

$$r_1 = (\beta_s - \beta_a)/(\beta_s + \beta_a) \tag{1}$$

with $\beta = n^2/(n^2-\alpha^2)^{1/2}$ for polarization parallel (‖), and with $\beta = (n^2-\alpha^2)^{-1/2}$ for polarization perpendicular ($\perp$) to the plane of incidence, respectively. Snell's invariant α is defined by $\alpha = n\sin\theta$. According to Eq. (1), r_1 depends on n_s with α as a parameter that can be adjusted by varying the angle of incidence. In a gaseous medium the variation of n_s can be assumed to be very small with respect to unity. Therefore the relation between the reflectivity r_1^2 and n_s can be linearized in good approximation. We linearize r_1^2 with respect to its value for a uniform reference state, indicated by subscript 0

$$r_1^2 = r_{10}^2(1 + S_1\Delta n) \tag{2}$$

with $\Delta n = n_s - n_0$. The sensitivity S_1 is defined as follows:

$$S_1 = \frac{2}{r_{10}}\left(\frac{\mathrm{d}r_1}{\mathrm{d}n_s}\right)_0 = \frac{4\beta_a}{\beta_0^2 - \beta_a^2}\left(\frac{\mathrm{d}\beta}{\mathrm{d}n}\right)_0 \tag{3}$$

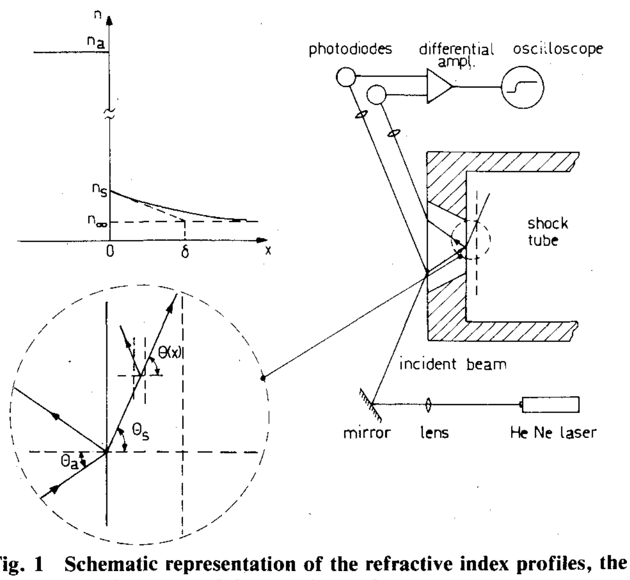

Fig. 1 Schematic representation of the refractive index profiles, the light wave trajectory, and the experimental setup.